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# Factorization Method and the Supersymmetric Monopole Harmonics

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## ABSTRACT

We use the general  $N = 1$  supersymmetric formulation of one dimensional sigma models on non trivial manifolds and its subsequent quantization to formulate the classical and quantum dynamics of the  $N = 2$  supersymmetric charged particle moving on a sphere in the field of a monopole. The factorization method is accommodated with the general covariance and it is used to integrate the corresponding system.

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# 1 Introduction

The factorization method was first used by Schrödinger <sup>[1]</sup> to diagonalize the harmonic oscillator. It may work when the Hamiltonian of the system can be cast as a product of two operators:

$$H = AB , \tag{1 . 1}$$

a c-number can also be added to the above operator.

Most systems treated by this method are one dimensional systems <sup>[2] [3]</sup>, and it has been also considered for other situations<sup>[4]</sup>.

Recently, Ferapontov and Veselov <sup>[5]</sup> used factorization to look for integrable Schrödinger operators with magnetic fields on two dimensional surfaces. In the following we will use only their solution for the monopole harmonics of Wu and Yang <sup>[6]</sup>.

We will show that factorization can be readily used for integrating the  $N = 2$  supersymmetric nonrelativistic quantum mechanics <sup>[7]</sup> of a particle with spin moving on a sphere in the field of a monopole placed at its center. This can be inferred from the fact that the supersymmetric charge of the classical  $\sigma$  - model quantizes as the Dirac operator<sup>[8] [9]</sup> on the manifold. Because the sphere is a two dimensional manifold, in a convenient basis the Dirac operators do not mix the components of the spinors and therefore the Hamiltonian of the system will be:

$$H = \frac{1}{2}(Q\bar{Q} + \bar{Q}Q) = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} ,$$

and because of this, the factorization method will be the natural method to integrate  $H$ .

As the factorization method is very simple we will replay some of its features here, in order to make the following computations easier to follow. Therefore along with (1 . 1), one considers the reverse order product:

$$\tilde{H} = BA .$$

Then  $\tilde{H}$  and  $H$  have the same non zero eigenvalues. Indeed, if  $\lambda$  is an eigenvalue ( $\lambda \neq 0$ ) of  $\tilde{H}$ :

$$\tilde{H} | \tilde{\Psi}_\lambda > = \lambda | \tilde{\Psi}_\lambda > ,$$

then

$$| \Psi_\lambda \rangle = A | \tilde{\Psi}_\lambda \rangle ,$$

obeys:

$$H | \Psi_\lambda \rangle = \lambda | \Psi_\lambda \rangle .$$

For the harmonic oscillator,

$$H = (a^\dagger a + 1/2) ,$$

with

$$[a, a^\dagger] = 1 ,$$

introduce:

$$\tilde{H} = (aa^\dagger + 1/2) = (a^\dagger a + 3/2) .$$

The operators  $H$  and  $\tilde{H}$  are positive operators, and it follows that the vector  $| 0 \rangle$ ,

$$a | 0 \rangle = 0$$

is an eigenvector of  $H$ , with  $\lambda = 1/2$ , and of  $\tilde{H}$ , with  $\lambda = 3/2$ , and therefore

$$a^\dagger | 0 \rangle ,$$

is an eigenvector of  $H$ , corresponding to  $\lambda = 3/2$ . The procedure can be continued:

$$\tilde{\tilde{H}} = H^{(2)} = \tilde{H} + 1 = H + 2 .$$

At the  $n$ -th step

$$H^{(n)} = H + n ,$$

with  $| 0 \rangle$  being an eigenvector of  $H^{(n)}$  corresponding to

$$\lambda = n + 1/2 .$$

Then  $H$ , has the same eigenvalue and the corresponding eigenvector is:

$$| \Psi_\lambda \rangle = (a^\dagger)^n | 0 \rangle .$$

We emphasize that the factorization method involves three steps: First, write the Hamiltonian in a factorized way; second, use a trick to cast  $\tilde{H}$  in a form similar to that of  $H$  (i.e. move the destruction operators to the right in the previous example). The third step is

to find the necessary recurrence procedure that yields all the eigenvalues and eigenvectors. This is also the pattern we are going to follow.

In Section 2 we reformulate the method of reference <sup>[5]</sup> for the scalar wave function in the einbein formalism. We show that the recurrence relations are obtained as a consequence of the requirement that the reverse order product  $BA$  has the same covariance properties acting on scalar wave vectors as the original product  $AB$ . On the sphere this happens because the spin connection associated with the  $U(1)$  gauge group in the tangent space can be chosen to be proportional to the gauge connection corresponding to the monopole at its center<sup>[10]</sup>, the proportionality factor being exactly the number entering into the Dirac quantization condition.

In Section 3 we formulate the maximal classical supersymmetric action associated with the motion of a charge on a sphere, in the field of a monopole located at its center. The same action can be obtained by considering the supersymmetric one dimensional  $SU(2)/U(1)$  nonlinear model and coupling it with the electromagnetic field through a gauge potential equal with the connection, generated by the nonlinear transformation law<sup>[11]</sup>. The equivalence of these two formulations stems from the aforementioned property: the spin connection and the gauge connection are proportional. Alternatively, the same result follows by restricting the  $N = 1$  supersymmetric action corresponding to a charge in the field of a monopole through Lagrange super-multipliers <sup>[12]</sup> <sup>[13]</sup>. However, we prefer to obtain our model from the existing general superspace  $\sigma$ -model actions <sup>[14]</sup> <sup>[15]</sup>, by fixing the field content, and choosing the appropriate background. In this way we can follow the natural way of solving, models of this sort. We will need only two real anticommuting classical degrees of freedom for the formulation of the classical supersymmetric action. After quantization these real anticommuting degrees of freedom become the gamma matrices. This suggests that the two bosonic degrees of freedom combine with the anticommuting ones in two type  $B$  superfields. Then we fix the appropriate background in the most general renormalizable  $N = 1$  type  $B$  superspace  $\sigma$ -model action<sup>[14]</sup> <sup>[15]</sup>. The quantization of the general superspace  $\sigma$ -models produces the component approach to supersymmetric quantum mechanics, the general formulation of which, was pursued in <sup>[16]</sup>. To quantize our model we use the procedure of <sup>[17]</sup> <sup>[18]</sup> in which the reparametrization covariance can be obtained with the help of the appropriately defined Noether supercharge.

Section 4 is devoted to the diagonalization of this quantum system, while in Section 5 we give some concluding remarks.

## 2 Factorization Method

The Hamiltonian for the motion of a charge  $e$ , on a sphere, in the field of a monopole of strength  $g$ , localized in the center of the sphere is:

$$H_N = -\frac{1}{4}g^{ab}\nabla_a^{(N)}\nabla_b^{(N)} , \quad (2.1)$$

where  $g^{ab}$  is the inverse metric on a sphere:

$$ds^2 = g_{ab}dx^a dx^b = R^2 \sin^2 \theta d^2 \phi + R^2 d^2 \theta , \quad (2.2)$$

and the covariant derivatives are:

$$\nabla_b^{(N)} = \partial_b - iA_b^{(N)} , \quad (2.3)$$

and:

$$\nabla_a^{(N)}\nabla_b^{(N)} = \partial_a\nabla_b^{(N)} - iA_a^{(N)}\nabla_b^{(N)} - \Gamma_{ab}^c\nabla_c^{(N)} , \quad (2.4)$$

where  $\partial_a$  are the derivatives with respect to the coordinates in a patch on the sphere. The gauge connection  $A_a^N$  is:

$$A_\phi = \frac{(eg)}{R \sin \theta} (\pm 1 - \cos \theta) , \quad (2.5)$$

where  $\theta < \pi$  for the upper sign and  $\theta > 0$  for the lower sign.  $\Gamma_{ab}^c$  is the standard Cristoffel connection for a sphere.

The desired factorization of (2.1) will be obtained by introducing the stereographic projection:

$$\cos \theta = 1 - \frac{2}{1 + \frac{x^2 + y^2}{4R^2}} , \quad (2.6)$$

$$\tan \phi = \frac{y}{x} . \quad (2.7)$$

Defining the complex coordinate:

$$z = x + iy , \quad (2.8)$$

the only nonvanishing components of  $g^{ab}$ , are  $g^{z\bar{z}} = g^{\bar{z}z}$  with:

$$g^{z\bar{z}} = 2h^{-2} = 2\left(1 + \frac{z\bar{z}}{4R^2}\right) . \quad (2.9)$$

Now, in the new coordinates the Hamiltonian is:

$$H_N = -\frac{1}{2}g^{z\bar{z}}\nabla_{\bar{z}}^{(N)}\nabla_z^{(N)} + \frac{N}{4R^2} , \quad (2.10)$$

where

$$\nabla_z^{(N)} = \partial_z - iA_z^{(N)} , \quad (2.11)$$

$$\nabla_{\bar{z}}^{(N)} = \partial_{\bar{z}} - iA_{\bar{z}}^{(N)} , \quad (2.12)$$

and

$$A_z^{(N)} = \frac{1}{2}(A_1^{(N)} - iA_2^{(N)}) = -iN\partial_z \ln h , \quad A_{\bar{z}}^{(N)} = A_z^{(N)*} . \quad (2.13)$$

The Cristoffel connection does not appear in (2.10) anymore, because its only nonvanishing components are  $\Gamma_{zz}^z$  and  $\Gamma_{\bar{z}\bar{z}}^{\bar{z}}$ . In (2.10) we have used the relation:

$$g^{z\bar{z}}[\nabla_z^{(N)}, \nabla_{\bar{z}}^{(N)}] = -\frac{N}{R^2} , \quad (2.14)$$

in order to exhibit the “destruction operators”  $\nabla_z^{(N)}$  to the right and the “creation operators”  $\nabla_{\bar{z}}^{(N)}$  to the left. The eigenvalues of  $H_N$  are defined by the zero modes of the operators  $\nabla_z^{(K)}$ . These zero modes are related by the index theorem to topological properties of the manifold, see reference [5]. Thus we may regard the present method of integrating  $H_N$  as a topological one.

In our case the wave vector is a scalar under the reparametrizations of the manifold. In a more general setting we might consider different assignments of spinorial (tensorial) properties of the wave vector. This is the case when one deals with the motion a charged spin one half particle on a sphere, when the wave function is a spinor. Then, for a manifestly covariant approach, the appropriate language is the vielbein formalism of general relativity. This is not actually necessary for the case of the scalar wave functions. However, as we will show, even in this case, it allows one to avoid the use of the correct, but rather artificial, similarity transformations [5] in establishing the recurrence relations necessary to apply the factorization method.

Let us now introduce the ein-beins for our complex manifold:

$$g_{z\bar{z}} = e_z^+ e_{\bar{z}}^- \eta_{+-} , \quad (2.15)$$

where  $\eta_{+-}$  is the metric in tangent space in an appropriate basis:

$$\eta_{+-} = \eta_{-+} = \frac{1}{2} , \quad \eta^{+-} = \eta^{-+} = 2 . \quad (2.16)$$

We have:

$$e_z^+ = e_{\bar{z}}^- = h , \quad e_z^- = e_{\bar{z}}^+ = 0 . \quad (2.17)$$

Using the constant covariance of the ein-bein:

$$\nabla_a e_b^\alpha = \partial_a e_b^\alpha + \omega_a{}^\alpha{}_\beta e_b^\beta - \Gamma_{ab}^c e_c^\alpha = 0 , \quad (2.18)$$

(2.10) becomes:

$$H_N = -\mathcal{D}_-^{(N)} \mathcal{D}_+^{(N)} + \frac{N}{4R^2} , \quad (2.19)$$

where:

$$\mathcal{D}_-^{(N)} = e_{\bar{z}}^{\bar{z}} (\nabla_{\bar{z}}^{(N)} + \omega_{z+}{}^+ ) , \quad (2.20)$$

and

$$\mathcal{D}_+^{(N)} = e_+^z \nabla_z^{(N)} . \quad (2.21)$$

Above, one has:

$$e_+^z = e_{\bar{z}}^{\bar{z}} = h^{-1} , \quad (2.22)$$

and the nonvanishing components of the spin connection are:

$$\omega_z{}^+{}_{+} = -\omega_{z+}{}^+ = \partial_z \ln h , \quad (2.23)$$

$$\omega_{\bar{z}}{}^+{}_{+} = -\omega_{\bar{z}+}{}^+ = -\partial_{\bar{z}} \ln h . \quad (2.24)$$

Consider now the tilde of (2.19) :

$$\tilde{H}_N = -[\mathcal{D}_+^{(N)}]_{nc} [\mathcal{D}_-^{(N)}]_{nc} + \frac{N}{4R^2} , \quad (2.25)$$

While (2.19) is manifestly generally covariant ( assuming that the wave vector is a world scalar), (2.25) is not manifestly so. By reversing the order of covariant derivatives, the spin connection terms in the covariant derivatives do not match the tensor properties of the terms ahead of them. This is why in considering the reverse order product above, we appended the  $nc$  index to the covariant derivatives, even if their definition (2.20), (2.21) did not change.

In order to obtain a recurrence relation one would expect (as explained in the Introduction) the operator  $\tilde{H}_N$  to be a scalar and act on a scalar wave function. However, then  $\tilde{H}_N$  is not manifestly covariant. The manifest covariance of  $\tilde{H}_N$  is restored by noting that the gauge connection and the spin connection can be chosen to be proportional <sup>[10]</sup>:

$$A_a^{(N)} = N\omega_a , \quad (2.26)$$

where  $N = eg$  appears in the Dirac quantization condition. Therefore we have the following identity:

$$\begin{aligned} [\mathcal{D}_+^{(N)}]_{nc} [\mathcal{D}_-^{(N)}]_{nc} &= e_+^z \nabla_z^{(N)} e_-^{\bar{z}} (\nabla_{\bar{z}}^{(N)} + \omega_{z+}^+) = \\ e_+^z (\nabla_z^{(N+1)} + \omega_{z-}^-) e_-^{\bar{z}} \nabla_{\bar{z}}^{(N+1)} &= \mathcal{D}_+^{(N+1)} \mathcal{D}_-^{(N+1)} . \end{aligned} \quad (2.27)$$

Here the expression of  $\mathcal{D}_+^{(N+1)}$ ,  $\mathcal{D}_-^{(N+1)}$ , can be read in the above formula and the the expression to the right, above, is fully covariant. Substituting this in  $\tilde{H}_N$  and moving the destruction operators to the right with the help of (2.14) we have:

$$\tilde{H}_N = H_N^{(1)} = H_{N+1} + \frac{(2N+1)}{4R^2} . \quad (2.28)$$

Therefore the next eigenvalue of  $H_N$  is  $\frac{3N+2}{4R^2}$ . The corresponding eigenvector can be obtained from that of  $\tilde{H}_N$ , given by the condition

$$\nabla_z^{(N+1)} \tilde{\Psi}_1 = 0 . \quad (2.29)$$

One has:

$$\Psi_1 = \mathcal{D}_-^{(N)} \tilde{\Psi}_1 , \quad (2.30)$$

with  $\mathcal{D}_-^{(N)}$  defined by (2.20). Even if we deal with two covariant problems, that of  $H_N$  and that of  $\tilde{H}_N$ , the connection between the two sets of eigenvectors is not generally covariant.

The procedure described above can be continued, and at the  $l$ -th step we get

$$H_N^{(l)} = H_{N+l} + \frac{[(2N-1) + l(l+1)]}{4R^2} , \quad (2.31)$$

with the eigenvalue

$$\lambda_l = \frac{1}{4R^2} [(2l+1)N + l(l+1)] , \quad (2.32)$$



and the corresponding eigenvector of  $H_N$

$$\Psi_l = \mathcal{D}_-^{(N)} \dots \mathcal{D}_-^{(N+l-1)} \tilde{\Psi}_l, \quad (2.33)$$

where  $\mathcal{D}_-^{(P)}$ , was defined in (2.20).  $\tilde{\Psi}_l$ , is the solution of

$$\nabla_z^{(N+l)} \tilde{\Psi}_l = 0. \quad (2.34)$$

The multiplicity of the state  $\Psi_l$  is obtained from the condition of finite norm of the states:

$$\int \frac{dz d\bar{z}}{2} h^2 |\tilde{\Psi}_l|^2 < \infty, \quad (2.35)$$

Indeed, the solution of (2.34) is:

$$\tilde{\Psi}_l = h^{N+l} f(\bar{z}), \quad (2.36)$$

Where  $f(\bar{z})$  is an arbitrary polynomial of degree  $\leq 2(N+l)$ , making the degeneracy of the corresponding state  $2(N+l)+1$ . From the asymptotic behaviour of (2.33) in the radial variable, one sees that there are potential problems with normalizability of such states. However, as we checked on examples, due to cancelations of unwanted terms the vector  $\Psi_l$  is normalizable. We point out that this result is valid for integer  $2N \geq 0$ . Otherwise ( $N < 0$ ) the creation and annihilation operators must be interchanged.

Hence imposing the manifest general covariance of  $\tilde{H}_N$  led us to rederive the recurrence relations <sup>[5]</sup> necessary in order to completely integrate the Hamiltonian  $H_N$ . In the next section using the canonical quantization we will obtain the Hamiltonian for the supersymmetric particle which will be subsequently diagonalized by factorization method.

### 3 $N = 2$ Supersymmetric Quantum Mechanics on a Sphere

We will approach the supersymmetrization of a given bosonic action in the following way. Given the target manifold (whose local coordinates are the bosonic fields which appear in the formulation of the 1-dimensional  $\sigma$ -model of the system), we look at the dimensionality of the Clifford algebra supported by the tangent space, for the sphere this is two. Therefore in the present case the  $\Gamma$ -matrices will be hermitian matrices and therefore can be obtained from the quantization of the two real anticommuting degrees of freedom. Thus the minimal

content of the fields realizing the representation of the supersymmetry algebra will be two real bosonic (the coordinate on the target manifold ) and two real anticommuting degrees of freedom. We can fit this degrees of freedom in two type- $B$  superfields. Because the tangent space is two dimensional and the supersymmetry charges must be constructed with the help of the  $\Gamma$ -matrices, one expects the maximal supersymmetry allowable for the system to be  $N = 2$ . Thus with the help of two type- $B$  superfields we must formulate an  $N = 2$  supersymmetric action. This is automatic since our target space manifold admits a complex structure. The above argument is somehow circular because one formulates the problem on the basis of the outcome of the quantization procedure.

Therefore with the help of the metric  $g_{ab}$  , the gauge connection  $A_a^{(N)}$  and the type- $B$  superfield  $X^a(x, \theta)$  , we construct the following  $N = 1$ , 1-dimensional sigma-model <sup>[14] [15]</sup> :

$$S = -i \int dt d\theta \left\{ \frac{g_{z\bar{z}}}{2} (DX^z \dot{X}^{\bar{z}} + DX^{\bar{z}} \dot{X}^z) + A_z DX^z + A_{\bar{z}} DX^{\bar{z}} \right\} . \quad (3.1)$$

Here

$$D = \frac{\partial}{\partial \theta} + i\theta \frac{d}{dt} , \quad (3.2)$$

is the covariant supersymmetric derivative. The superfields  $X^z$  and  $X^{\bar{z}}$ , are connected through  $X^{\bar{z}} = (X^z)^\dagger$  and have the components:

$$z = X^z|_{\theta=0} , \quad \lambda^z = (DX^z)|_{\theta=0} , \quad (3.3)$$

with:

$$\lambda^{z\dagger} = -\lambda^{\bar{z}} , \quad (3.4)$$

and

$$\int d\theta \{ \dots \} = D\{ \dots \}|_{\theta=0} . \quad (3.5)$$

The action (3.1) is invariant under the supersymmetry transformations

$$\delta_\epsilon X^{\bar{z}} = \epsilon Q X^{\bar{z}} , \quad (3.6)$$

$$\delta_\epsilon X^z = \epsilon Q X^z , \quad (3.7)$$

where the supersymmetry shift operator  $Q$  is

$$Q = \frac{\partial}{\partial \theta} - i\theta \frac{d}{dt} , \quad (3.8)$$

with

$$Q^2 = -i \frac{d}{dt} , \quad \{Q, D\} = 0 , \quad (3.9)$$

As mentioned before, because of the fact that the target space manifold is complex we will automatically have a second supersymmetry

$$\delta_\eta X^{\bar{z}} = -i\eta DX^{\bar{z}} , \quad (3.10)$$

$$\delta_\eta X^z = i\eta DX^z . \quad (3.11)$$

The two supersymmetry transformations above can be combined in one complex supersymmetry which in component fields  $z, \bar{z}, \lambda^z, \lambda^{\bar{z}}$  takes the form:

$$\delta z = \delta\mu \lambda^z , \quad \delta \lambda^z = i\delta\bar{\mu} \dot{z} , \quad (3.12)$$

and

$$\delta \bar{z} = \delta\bar{\mu} \lambda^{\bar{z}} , \quad \delta \lambda^{\bar{z}} = i\delta\mu \dot{\bar{z}} . \quad (3.13)$$

Where

$$\delta\mu^\dagger = \delta\bar{\mu} . \quad (3.14)$$

The action (3.1) can also be written in terms of component fields, and the Lagrangian is:

$$\mathcal{L} = g_{z\bar{z}} \dot{z} \dot{\bar{z}} + i \frac{g_{z\bar{z}}}{2} (\lambda^z D \lambda^{\bar{z}} + \lambda^{\bar{z}} D \lambda^z) - i F_{z\bar{z}}^{(N)} \lambda^z \lambda^{\bar{z}} + A_z^{(N)} \dot{z} + A_{\bar{z}}^{(N)} \dot{\bar{z}} , \quad (3.15)$$

where

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z , \quad (3.16)$$

and

$$D \lambda^z = \dot{\lambda}^z + \dot{z} \Gamma_{zz}^z \lambda^z , \quad D \lambda^{\bar{z}} = \dot{\lambda}^{\bar{z}} + \dot{\bar{z}} \Gamma_{\bar{z}\bar{z}}^{\bar{z}} \lambda^{\bar{z}} , \quad (3.17)$$

where  $\Gamma_{zz}^z$  and  $\Gamma_{\bar{z}\bar{z}}^{\bar{z}}$  are the only nonvanishing components of the Cristoffel connection for the sphere. (3.15) is invariant under (3.12) and (3.13) and by the Noether procedure one can deduce the corresponding supercharges:

$$\bar{Q} = -\lambda^z g_{z\bar{z}} \dot{\bar{z}} , \quad (3.18)$$

$$Q = \lambda^{\bar{z}} g_{\bar{z}z} \dot{z} . \quad (3.19)$$

From (3.15) one infers that one can go to the tangent space indices by making the redefinition :

$$\lambda^z = i e_+^z \lambda^+ , \quad (3.20)$$

$$\lambda^{\bar{z}} = i e_-^{\bar{z}} \lambda^- . \quad (3.21)$$

Then the fermions in (3 . 15) acquire a standard form:

$$\mathcal{L} = g_{z\bar{z}}\dot{z}\dot{\bar{z}} - \frac{i}{4}(\lambda^+\dot{\lambda}^- + \lambda^-\dot{\lambda}^+) + \dots , \quad (3 . 22)$$

and one concludes that  $\lambda^\pm$  quantizes as the corresponding  $\frac{1}{\sqrt{2}}\Gamma^\pm$  matrices:

$$\Gamma^{(+)} = \Gamma^{(-)\dagger} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} , \quad (3 . 23)$$

The canonical conjugate momentum is given by:

$$P_z = g_{z\bar{z}}\dot{\bar{z}} + \frac{i}{2}(\omega_z)_{\alpha\beta}\lambda^\alpha\lambda^\beta + A_z^{(N)} . \quad (3 . 24)$$

With the notation:

$$\Pi_z = g_{z\bar{z}}\dot{\bar{z}} = P_z - \frac{i}{2}(\omega_z)_{\alpha\beta}\lambda^\alpha\lambda^\beta - A_z^{(N)} , \quad (3 . 25)$$

the corresponding Noether charge can be written :

$$Q = \lambda^z\Pi_z = ie_+^z\lambda^+(P_z - \frac{i}{2}(\omega_z)_{\alpha\beta}\lambda^\alpha\lambda^\beta - A_z^{(N)}) . \quad (3 . 26)$$

Let us now quantize the supercharge  $Q$  <sup>[17] [18]</sup>. As mentioned before  $\lambda^\pm$  quantizes as the corresponding  $\frac{1}{\sqrt{2}}\Gamma^\pm$ , further  $P_z$  goes into  $\frac{1}{i}\frac{\partial}{\partial z}$ , therefore in order to maintain the general covariance under quantization we can take the minimal  $Q$ :

$$Q = \frac{\Gamma^+}{\sqrt{2}}e_+^z(\frac{1}{i})\nabla_z^{(N)} , \quad (3 . 27)$$

where  $\nabla_z^{(N)}$ , is the covariant derivative, on the spinor wave function:

$$\nabla_z^{(N)} = \partial_z - iA_z^{(N)} - \frac{1}{4}\omega_z^{\alpha\beta}\Gamma_{\alpha\beta} , \quad (3 . 28)$$

and the matrices  $\Gamma_{\alpha\beta}$  are

$$\Gamma_{\alpha\beta} = \frac{1}{2}[\Gamma_\alpha, \Gamma_\beta] , \quad (3 . 29)$$

and

$$\{\Gamma_\alpha, \Gamma_\beta\} = 2\eta_{\alpha\beta} , \quad (3 . 30)$$

Once we have the quantum expression for  $Q$  we can define

$$\bar{Q} = Q^\dagger , \quad (3 . 31)$$

with respect to the scalar product corresponding to (2 . 35), where of course now the scalar wave vector is replaced by the two component wave function. The adjoint of  $Q$  is:

$$Q^\dagger = -\frac{\Gamma^-}{\sqrt{2}} e^{\bar{z}} \nabla_{\bar{z}}^{(N)} , \quad (3 . 32)$$

with

$$\nabla_{\bar{z}}^{(N)} = \partial_{\bar{z}} - iA_{\bar{z}}^{(N)} - \frac{1}{4}\omega_{\bar{z}}^{\alpha\beta}\Gamma_{\alpha\beta} , \quad (3 . 33)$$

The  $Q$  and  $\bar{Q}$  so defined obey automatically

$$Q^2 = \bar{Q}^2 = 0 , \quad (3 . 34)$$

because they contain the matrices  $\Gamma^\pm$  in their definition. Defining the quantum Hamiltonian by

$$H = \frac{1}{2}(Q\bar{Q} + \bar{Q}Q) , \quad (3 . 35)$$

$H$  commutes automatically with the supercharges  $Q$  and  $\bar{Q}$ . Thus the quantum ordering in  $H$  is completely fixed by supersymmetry and reparametrization covariance. The expression for  $H$  is also covariant under the reparametrizations of the manifold and the system has  $N = 2$  supersymmetry.

Finally it might be worth mentioning that by this procedure  $H$  is defined from the supersymmetry algebra (3 . 35) without any recourse to the standard ways of defining it.

## 4 Supersymmetric Monopole Harmonics

Diagonalization of the Hamiltonian (3 . 35) follows now in a rather simple way. Using the constant covariance of the  $\Gamma$ -matrices with respect to vector and spinor indices (3 . 35) can be cast in the following form

$$H = -\frac{1}{4}e_+^z e_-^{\bar{z}} [\Gamma^+ \Gamma^- \nabla_z^{(N)} \nabla_{\bar{z}}^{(N)} + \Gamma^- \Gamma^+ \nabla_{\bar{z}}^{(N)} \nabla_z^{(N)}] , \quad (4 . 1)$$

the products  $\Gamma^+ \Gamma^-$  and  $\Gamma^- \Gamma^+$  are scalars under local rotations in the tangent plane. Moreover they are projectors:

$$\Gamma^{(+)}\Gamma^{(-)} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} , \quad \Gamma^{(-)}\Gamma^{(+)} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} , \quad (4 . 2)$$

Therefore  $H$  is a sum of factorized terms. We recognize that the operators multiplying the products  $\Gamma^+\Gamma^-$  and  $\Gamma^-\Gamma^+$  are connected with the Laplacian appearing in (2 . 1) with a modified spin connection, due to the nonzero spin of the wave function. In fact we have:

$$H = [H_{(N+\frac{1}{2})} + \frac{(N+\frac{1}{2})}{4R^2}] \frac{\Gamma^{(+)}\Gamma^{(-)}}{4} + [H_{(N-\frac{1}{2})} - \frac{(N-\frac{1}{2})}{4R^2}] \frac{\Gamma^{(-)}\Gamma^{(+)}}{4}, \quad (4 . 3)$$

This is due to the fact that the spinor components transform with an effective charge  $\pm\frac{1}{2}$  under local rotations. As remarked before one can absorb this ‘‘Lorentz charge’’ in the gauge connection leading to a modification of the effective charge of the corresponding components. Using (2 . 28), it is easy to show that:

$$\tilde{H}_{(N-\frac{1}{2})} - \frac{(N-\frac{1}{2})}{4R^2} = H_{(N+\frac{1}{2})} + \frac{(N+\frac{1}{2})}{4R^2}, \quad (4 . 4)$$

However, by the same procedure  $\tilde{H}_{N+\frac{1}{2}}$  gets connected with  $H_{N+\frac{3}{2}}$ . The connection between  $\tilde{H}_{N+\frac{1}{2}}$  and  $H_{N-\frac{1}{2}}$  is the result of a different factorization of  $H_N$  in Section 2. From (2 . 1) one has:

$$H_N = -\mathcal{D}_+^{(N)}\mathcal{D}_-^{(N)} - \frac{N}{4R^2}, \quad (4 . 5)$$

where

$$\mathcal{D}_+^{(N)} = e_+^z (\nabla_z^{(N)} + \omega_{z-}^-), \quad (4 . 6)$$

and

$$\mathcal{D}_-^{(N)} = e_-^{\bar{z}} \nabla_{\bar{z}}^{(N)}, \quad (4 . 7)$$

therefore with respect to this factorization we have:

$$\begin{aligned} \tilde{H}_N &= -[\mathcal{D}_-^{(N)}]_{nc}[\mathcal{D}_+^{(N)}]_{nc} - \frac{N}{4R^2} = \\ &= -\mathcal{D}_-^{(N-1)}\mathcal{D}_+^{(N-1)} - \frac{N}{4R^2} = H_{N-1} - \frac{2N-1}{4R^2}, \end{aligned} \quad (4 . 8)$$

and using the identities just derived one has:

$$\tilde{H}_{(N+\frac{1}{2})} + \frac{(N+\frac{1}{2})}{4R^2} = H_{(N-\frac{1}{2})} - \frac{(N-\frac{1}{2})}{4R^2}, \quad (4 . 9)$$

We prefer to start with the eigenfunctions of  $H_{N+\frac{1}{2}}$ , because this operator appears in a manifestly positive definite combination in the Hamiltonian.

Therefore given the nonzero eigenvalue eigenvector  $\Psi$  of  $H_{N+\frac{1}{2}} + (N+\frac{1}{2})$  we obtain the corresponding eigenvector of  $H_{N-\frac{1}{2}} - (N-\frac{1}{2})$  by taking:

$$\mathcal{D}_-^{(N-\frac{1}{2})}\Psi, \quad (4 . 10)$$

with  $\mathcal{D}_-^{(N-\frac{1}{2})}$ , from (2 . 20) ; From (4 . 3) the eigenvalues of  $H$  are:

$$E_l = \frac{1}{4R^2}(l+1)[l+2N+1] , \quad (4 . 11)$$

with the eigenvector being given by

$$\Psi_l = \begin{pmatrix} \mathcal{D}_-^{(N+\frac{1}{2})} \dots \mathcal{D}_-^{(N+l-\frac{1}{2})} \tilde{\Psi}_l \\ \mathcal{D}_-^{(N-\frac{1}{2})} \mathcal{D}_-^{(N+\frac{1}{2})} \dots \mathcal{D}_-^{(N+l-\frac{1}{2})} \tilde{\Psi}_l \end{pmatrix} . \quad (4 . 12)$$

Therefore we have found the eigenvalues and eigenvectors of the  $N = 2$  supersymmetric spinning particle moving on a sphere in the field of a monopole. In spherical coordinates they appear in <sup>[19]</sup>.

## 5 Conclusions

Restating the main result, we have supersymmetrized and solved the motion of a charge on a sphere in the field of a monopole at its center.

The factorization method appears to be the natural way to solve this problem,. This is because in order to formulate a supersymmetric problem one is basically compelled to use the complex structure of the target space manifold. The quantization scheme for the problem is manifestly taking into account the reparametrizations of the manifold therefore it is covariant with respect to this reparametrizations. One should remember that one is dealing basically with the algebra of the angular momentum and it is quite interesting that following the manifest symmetries of the problem one is led to an alternative integration method. In this context, even if algebraic, this method appears somewhat strange, albeit natural.

One should also stress that the degeneracies of the levels are all finite and it is well known that we deal with a regularization of the Landau electrons. Taking the the limit  $R^2, N \rightarrow \infty$  (with  $\frac{N}{R^2}$  fixed) one obtains the infinitely degenerate states of a planar electron in a constant magnetic field.

In a rather different context, the eigenfunctions obtained in this paper may help to define an alternative harmonic superspace <sup>[20]</sup>.

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